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Anomalous diffusion in dynamical systems: Transport coefficients of all order

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The theory of Ruelle's zeta function [*Thermodynamic Formalism* (Addison-Wesley, Reading, MA, 1978)] is extended to describe anomalous transport induced by dynamical chaos. It is shown that $P(q)$ for the generating function of the displacement may not exist for supradiffusive processes, and that the difficulty may be overcome by the introduction of a two-parameter function $P(\beta,q)$. We present two exactly solvable examples of anomalous diffusion induced by intermittency, to which our method is applied.

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I. INTRODUCTION

A transport process such as diffusion is a most conspicuous phenomenon in spatially extended chaotic systems. Examples include fluid mixing, high-temperature plasma, and celestial mechanics. Over the last few years, deterministic diffusion has been studied from the point of view of dynamical systems. The transport coefficient and the decay rates of relaxation have been successfully related to dynamical quantities of deterministic chaos such as the Lyapunov exponent, the Kolmogorov-Sinai entropy, and the Ruelle resonances [1].

Many model systems of deterministic diffusion display a chaotic "sea" that coexists with certain regular orbits in the phase space [2]. If a regular component is merely neutrally stable, a typical orbit can return and remain in its neighborhood for a very long time, with a pausing-time distribution displaying an algebraic long tail. Each such sojourn results in an episode of either ballistic or stagnant motion, depending on the nature of the regular component. Consequently, the square displacement after a time t may grow like $\langle[\mathbf{r}(t)-\mathbf{r}(0)]^2\rangle \sim t^\gamma$, $\gamma\neq 1$; then the diffusion coefficient does not exist ($\gamma>1$ in the ballistic case and $\gamma<1$ in the stagnant case). The effects may also be subtler when $\gamma=1$, but certain higher moments of $\mathbf{r}(t)-\mathbf{r}(0)$ behave anomalously, and higher-order transport coefficients such as the Burnett coefficient [3] are divergent. A systematic approach to anomalous diffusion should enable one to calculate the transport coefficients of arbitrarily high order. The aim of the present work is to propose such a general description.

A natural way to describe diffusion is to introduce a function $P(q)$ as follows [4,5]:

$$P(q)=\lim_{t\to\infty}\frac{1}{t}\ln\langle e^{q(\mathbf{r}(t)-\mathbf{r}(0))}\rangle \ , \tag{1}$$

where the average $\langle\ \rangle$ is performed with an invariant measure of the dynamical system. Then, in principle, the long-term behavior of all moments of $\mathbf{r}(t)-\mathbf{r}(0)$ could be deduced from $P(q)$. Let us define the kth-order transport coefficient as

$$\mathcal{B}_k=\frac{1}{(2k)!}\frac{d^{2k}P}{dq^{2k}}\bigg|_{q=0} \tag{2}$$

[we assume that the system is translationally symmetric, so that $P(q)=P(-q)$]; then the diffusion coefficient $\mathcal{D}$ and the Burnett coefficient $\mathcal{B}$ are given by

$$\begin{aligned}\mathcal{D}\equiv\mathcal{B}_1&=\lim_{t\to\infty}\frac{1}{2!}\langle[\mathbf{r}(t)-\mathbf{r}(0)]^2\rangle \ ,\\ \mathcal{B}\equiv\mathcal{B}_2&=\lim_{t\to\infty}\frac{1}{4!}\{\langle[\mathbf{r}(t)-\mathbf{r}(0)]^4\rangle\\ &\qquad -3\langle[\mathbf{r}(t)-\mathbf{r}(0)]^2\rangle^2\} \ .\end{aligned} \tag{3}$$

In a most general sense, we shall say that a diffusive process is *normal* if all $\mathcal{B}_k$ exist, and that it is *anomalous* otherwise. In other words, anomalous diffusion is identified to the nonanalyticity of $P(q)$ at $q=0$.

However, as we shall demonstrate below, for anomalous diffusion it is not guaranteed that $P(q)$ exists. If this happens to a chaotically diffusive state that is not isolated from a certain regular state, the latter will dominate the long-term behavior of the system, and hence determine $P(q)$. In this paper we introduce a *two-variable* function $P(\beta,q)$, a generalization of $P(q)$, where β helps to select the desired invariant measure to be used in computing transport coefficients. We shall illustrate our approach by two exactly solvable models of anomalous diffusion in intermittent dynamics. Those are perhaps the simplest situations of chaotic motion in the presence of a ballistic mode or a stagnant mode, respectively, so our conclusions drawn here may be considered as quite general. At the end, we shall apply our results to the interpretation of a recent numerical work on the Hamiltonian standard map.

II. THE ZETA FUNCTION

Let us consider a map $\tilde{f}(x)$ defined in a one-dimensional array of unit intervals (cf. Fig. 1), such that $\tilde{f}(1-x)=1-\tilde{f}(x)$ for $x\in[0,1]$ (reflection symmetry) and $\tilde{f}(k+x)=k+\tilde{f}(x)$ (translational symmetry). The unit interval [0,1] is divided into three subintervals, I_{-1}, I_0, and I_{+1}, such that $[\tilde{f}(x)]=i$ if $x\in I_i$. ($[x]$ denotes the integer part of x.) The displacement after n iterations starting at x is given by $r(n,x)=[\tilde{f}^{\,n}(x)]$. Because of the translational symmetry, the diffusion can be described by a reduced map $f(x)=\tilde{f}(x)$ (mod 1) defined on an elementary cell (the unit interval). In terms of an indicator function $I(x)=i$ if $x\in I_i$, the displacement in the original system is now expressed as $r(n,x)=\sum_{k=0}^{n-1}I(f^k(x))$.

According to the thermodynamic formalism of dynamical systems [6], we introduce a topological pressure that depends on two parameters, β and q:

$$P(\beta,q)=\lim_{n\to\infty}\frac{1}{n}\ln Z_n(\beta,q)\ , \tag{4a}$$

where $Z_n(\beta,q)$ is a partition function, given by a sum over all the cyclic orbits of period n of the map $f(x)$,

$$Z_n(\beta,q)=\sum_{f^n(x)=x}\exp\left[-\beta\sum_{k=0}^{n-1}\ln|f'(f^k(x))| + q\sum_{k=0}^{n-1}I(f^k(x))\right]. \tag{4b}$$

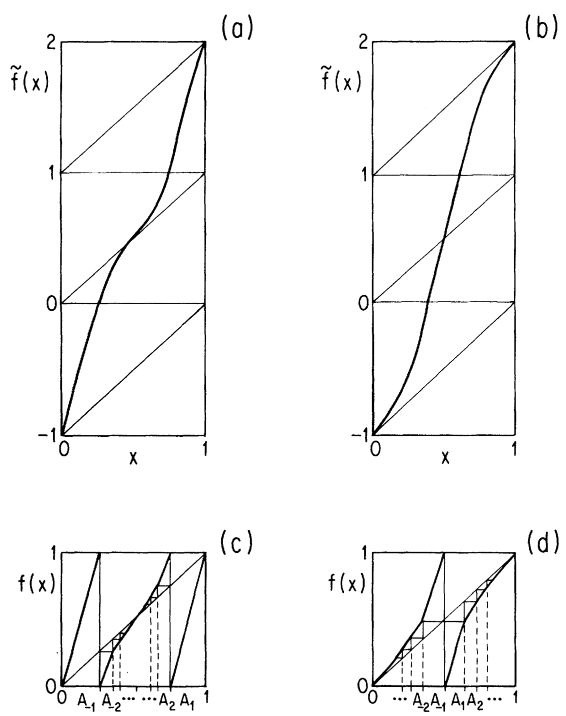

FIG. 1. (a) The GT map and (b) the GNZ map defined in the interval [0,1]. (c) The reduced GT map that is piecewise linearized in each of the doubly infinite intervals $A_{\pm k}$, $k=1,2,\ldots$. (d) Same as (c) for the reduced GNZ map, where the middle branch is neglected.

The $P(\beta,q)$ function, from which many diffusion properties can be derived (as we shall see below), is expressed solely in terms of the dynamical stretching factors of the periodic orbits, without *ad hoc* assumptions or probabilistic approximations.

We have $P(\beta,q)=-\beta F(\beta,q)$, where $F(\beta,q)$ is formally a free energy. $P(\beta,q=0)$ is the well-known Ruelle topological pressure of the map $f(x)$ [6]. The parameter β, formally an inverse temperature, selects an invariant measure of the system. In particular, $\beta=1$ corresponds to the natural measure of Sinai, Bowen, and Ruelle. The second parameter, q, is formally analogous to an external field. The partial derivatives of $P(\beta,q)$ with respect to q, at $q=0$, yield the *generalized* transport coefficients $\mathcal{B}_n(\beta)$, corresponding to the β-dependent invariant measure. Note that $P(\beta=1,q)=P(q)$ of Eq. (1). If we define a zeta function

$$\zeta(z,\beta,q)=\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}z^nZ_n(\beta,q)\right], \tag{5}$$

then $P(\beta,q)=\ln[1/z^*(\beta,q)]$, where $z^*(\beta,q)$ is the smallest singularity of $\zeta(z,\beta,q)$.

Consider orbits of length n such that

$$\frac{1}{n}\sum_{k=0}^{n-1}\ln|f'(f^k(x))|\sim\lambda,\quad \frac{1}{n}\sum_{k=0}^{n-1}I(f^k(x))\sim v\ , \tag{6}$$

where λ and v are the Lyapunov exponent and the displacement velocity of finite orbits. Suppose that the number of orbits with the same values of λ and v grows exponentially with the orbit's length n, as $\sim\exp[nS(\lambda,v)]$, where $S(\lambda,v)$ is an entropic function. Then, we have

$$P(\beta,q)=S(\langle\lambda\rangle,\langle v\rangle)-\langle\lambda\rangle\beta+\langle v\rangle q\ , \tag{7}$$

where $\langle\lambda\rangle$ and $\langle v\rangle$ are the *mean* values as functions of β and q. $S(\langle\lambda\rangle,\langle v\rangle)$ is hence a Legendre transform of $P(\beta,q)$.

Let us give a simple example. Consider the chaotic diffusion induced by the following map [7]:

$$\tilde{f}(x)=\begin{cases}7x & \text{if } 0\le x<\frac{2}{7}\\ 7(\frac{1}{2}-x)+\frac{1}{2} & \text{if } \frac{2}{7}\le x<\frac{5}{7}\\ 7(x-1)+1 & \text{if } \frac{5}{7}\le x\le 1\end{cases} \tag{8}$$

[with $\tilde{f}(x+k)=k+\tilde{f}(x)$]. The function $P(\beta,q)$ can be readily obtained, which gives

$$P(\beta,q)=\ln[3+4\cosh(q)]-\beta\ln 7\ . \tag{9}$$

Note that $P(\beta)=P(\beta,q=0)=(1-\beta)\ln 7$. Transport coefficients of all orders exist since $P(q)$ is an analytic function of q. Furthermore, from Eqs. (7) and (9), we have

$$\langle v\rangle=\frac{\partial P}{\partial q}=\frac{4\sinh(q)}{3+4\cosh(q)}\ . \tag{10}$$

In this case, $\langle v\rangle$ is independent of β. Therefore, the mean velocity is not zero if the "external field" q is present. It is a monotonically increasing function of q, converging to ±1 as $q\to\pm\infty$, respectively.

III. ANOMALOUS DIFFUSION

In what follows we shall calculate $P(\beta,q)$ explicitly for two exactly solvable models of anomalous diffusion. They are piecewise-linearized versions of the Geisel-Thomas (GT) map and the Geisel-Nierwetberg-Zacherl (GNZ) map [8] displayed in Figs. 1(a) and 1(b). The GT (GNZ) map is known to exhibit subdiffusive (supradiffusive) behavior due to a Pomeau-Manneville (PM) mechanism of intermittency. We shall show by these examples how the analyticity of $P(\beta,q)$ is broken because of anomalous effects of arbitrary order.

First consider the GT map [Fig. 1(a)]. Its reduced map $f(x)$ has a slope everywhere larger than 1, except at $x=\frac{1}{2}$. For $|x-\frac{1}{2}|\ll 1$, $f(x)$ has the following form:

$$f(x+\tfrac{1}{2})=\tfrac{1}{2}+x+c\,\mathrm{sgn}(x)|x|^{(1+1/\alpha)},\quad \alpha>0\ . \tag{11}$$

With the partition shown in Fig. 1(c), the unit interval is divided into a doubly infinite cells, A_k, $k=\pm 1,\pm 2,\ldots$. We can choose A_k such that

$$\Delta_k\equiv|A_{\pm k}|\sim[k^{-\alpha}-(k+1)^{-\alpha}]\underset{k\to\infty}{\sim}(\alpha/2)k^{-(\alpha+1)}\ , \tag{12}$$

$\sum_{k=1}^{\infty}\Delta_k=\frac{1}{2}$. The map is linearized in each cell, much like for the PM map of intermittency [9]. The slope of the map in the cell $A_{\pm k}$ is $s_k=|\Delta_{k-1}/\Delta_k|$, $k=1,2,\ldots$ ($\Delta_0=1$). The local stretching rate is

$$\lambda_k\equiv\ln s_k\underset{k\to\infty}{\sim}1/k\to 0\ .$$

For this simple model, the zeta function of Eq. (5) can be calculated exactly (see the Appendix), which yields

$$\zeta^{-1}(z,\beta,q)=(1-z)[1-2M_0(z,\beta)\cosh(q)]\ , \tag{13}$$

where $M_0(z,\beta)=\sum_{n=1}^{\infty}z^n\Delta_n^{\beta}$. As expected, there are two ways for $\zeta(z,\beta,q)$ to diverge. The simple singularity $z_0=1$ corresponds to the neutral fixed point, and the singularity z^*, given by

$$M_0(z^*,\beta)=\tfrac{1}{2}\mathrm{sech}(q)\ , \tag{14}$$

describes the chaotic diffusion for almost all initial conditions.

Which of the two singularities is the smallest one [and hence determines $P(\beta,q)$] depends on the parameters β and q. The (β,q) plan is divided into two regions (localized and diffusive state) [Fig. 2(a)]. Note that, with $q=0$, $P(\beta,q=0)$ is reduced to the $P(\beta)$ of the PM intermittency, with $P(\beta)\equiv 0$ for $1\le\beta$ [9]. In the (β,q) plan, the curve $\beta(q)$, which separates the localized state from the diffusive one, is given by inserting $z^*=1$ into Eq. (14). The curve thus obtained is an even function of q, with a unique minimum at $\beta=1$ [see Fig. 2(a)]. In other words, even for $\beta\ge 1$, when the localized state is dominant if $q=0$, the application of an "external field" ($q\neq 0$) may delocalize the system, and chaotic diffusion becomes prevalent.

Many properties of the diffusive dynamics can be obtained from Eqs. (12) and (14). For instance, the diffusion coefficient $\mathcal{D}$ is given by [10]

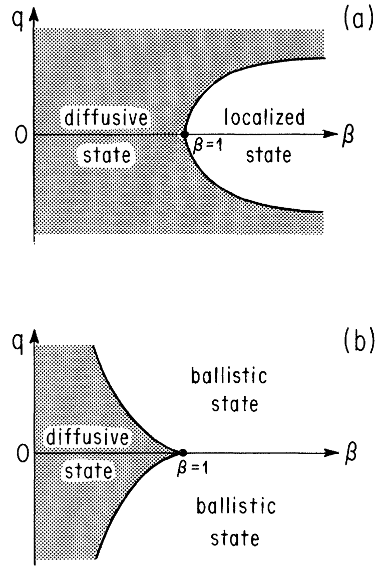

FIG. 2. The phase diagram in the (β,q) plan. (a) The GT map has two phases: diffusive and localized motion. (b) The GNZ map has three phases: one diffusive and two ballistic motions. The anomalous diffusion occurs along the phase-separation curve in both cases. Please see the text for more details.

$$\mathcal{D}=\frac{1}{4\sum_{n=1}^{\infty}n\Delta_n}=\begin{cases}0 & \text{if } \alpha\le 1\\ \text{finite} & \text{if } \alpha>1\ .\end{cases} \tag{15}$$

In fact, the transport coefficients of all orders, $\mathcal{B}_k$, can be explicitly calculated. Qualitatively, one can deduce from Eq. (14) that, in the case of $\alpha<1$, $P(\beta=1,\,q)\sim q^{2/\alpha}$ for $|q|\ll 1$. Thus, if $1/(m+1)<\alpha<1/m$,

$$\mathcal{B}_k=0,\quad k=1,2,\ldots,m,\quad\text{and }\mathcal{B}_{m+1}=\infty\ . \tag{16a}$$

And, if $1\le m<\alpha<m+1$, we have

$$P(\beta=1,\,q)=\sum_{k=1}^{m}\mathcal{B}_kq^{2k}+\mathcal{B}_\alpha q^\alpha+o(q^\alpha)\ ;$$

hence,

$$\mathcal{B}_k=\text{finite},\quad k=1,2,\ldots,m,\quad\text{and }\mathcal{B}_{m+1}=\infty\ . \tag{16b}$$

These conclusions are universal in the sense that they depend only on the exponent α of the original system. In particular, we emphasize that, when the diffusion coefficient $\mathcal{D}$ is finite, transport coefficients of sufficiently high order must diverge.

We now turn to the GNZ map [Fig. 1(b)]. As before, we reduce the problem to a circle map $f(x)$. In this case the middle branch of $f(x)$ is not essential, and is replaced by a vertical line at $x=\frac{1}{2}$; the remaining two branches are matched so that $f(x)$ is defined over the whole unit interval [see Fig. 1(d)]. The map $f(x)$ satisfies

$$f(x)=\begin{cases} x+cx^{1+1/\alpha} & \text{if } x\ll 1 \\ x-c(1-x)^{1+1/\alpha} & \text{if } 1-x\ll 1 \;, \end{cases} \quad (17)$$

for $0<\alpha$. The function $f(x)$ is linearized in each of the doubly infinite cells shown in Fig. 1(d), with $\Delta_k=|A_{\pm k}|$ satisfying Eq. (12).

For this map, the zeta function $\zeta(z,\beta,q)$ can again be explicitly obtained, which yields

$$\zeta^{-1}(z,\beta,q)=(1-ze^{-q})(1-ze^{+q})(1-M_+M_-)\;,$$

$$M_\pm=\sum_{n=1}^{\infty} z^n(2\Delta_n)^\beta e^{\pm qn}\;. \quad (18)$$

We have either $P(\beta,q)=\pm q$ of the ballistic motions (with constant velocity ± 1), or the nontrivial $P(\beta,q)=\ln(1/z^*)$ of chaotic diffusion, where z^* is given by

$$\sum_{n=1}^{\infty} z^{*n}(2\Delta_n)^\beta e^{qn}\sum_{m=1}^{\infty} z^{*m}(2\Delta_m)^\beta e^{-qm}=1\;. \quad (19)$$

All transport coefficients $\mathcal{B}_k$ can be computed from Eq. (19). For instance, the diffusion coefficient [11]

$$\mathcal{D}=\frac{\sum_{n=1}^{\infty} n^2\Delta_n-2\left[\sum_{n=1}^{\infty} n\Delta_n\right]^2}{2\sum_{n=1}^{\infty} n\Delta_n}$$

$$=\begin{cases}\text{finite} & \text{if } \alpha>2 \\ \infty & \text{if } \alpha\le 2\;. \end{cases} \quad (20)$$

It can be readily seen that as long as $q\neq 0$ and $\beta\geq 1$ no solution exists for Eq. (19). So the situation here is quite different from the previous case: letting $\beta=1$, we do not have a nontrivial singularity, and $P(q)$ must be that of a fixed point of the ballistic motion [12]. The (β,q) plan [Fig. 2(b)] is divided into two (ballistic and diffusive) phases, separated by a curve $\beta(q)$ given by inserting $z^*=\exp(\pm q)$ into Eq. (19). For $1-\beta\ll 1$, one has

$$[1-\beta(q)]\sim\begin{cases}|q| & \text{if } \alpha>1 \\ |q|^\alpha & \text{if } \alpha<1\;. \end{cases} \quad (21)$$

Therefore, one must first calculate the nontrivial $P(\beta,q)$ with $\beta<1$, and then study the limit $\beta\to 1^-$ afterwards. Let us expand $P(\beta,q)$ for $|q|\ll 1-\beta\ll 1$,

$$P(\beta,q)=P(\beta)+\sum_{k=1}^{\infty}\mathcal{B}_k(\beta)q^{2k}\;, \quad (22)$$

where $\mathcal{B}_k(\beta)$ are the generalized transport coefficients as function of β. From Eq. (19) one can show that as long as $\beta<1$ all the coefficients $\mathcal{B}_k(\beta)$ are finite, and hence the diffusion is normal. Moreover, in the limit $\beta\to 1^-$, in the case of $\alpha<1$,

$$\mathcal{B}_k(\beta)\underset{\beta\to 1^-}{\sim}(1-\beta)^{(1-2k)/\alpha}\to\infty\;; \quad (23a)$$

and in the case of $\alpha>1$, if $2m<\alpha<2(m+1)$, $m=0,1,2,\ldots$, then

$$\mathcal{B}_k(\beta)\underset{\beta\to 1^-}{\sim}\begin{cases}\text{const} & \text{if } k\le m \\ (1-\beta)^{(\alpha-2k)}\to\infty & \text{if } k>m\;. \end{cases} \quad (23b)$$

For instance, in the limit $\beta\to 1^-$, if $1<\alpha<2$, then $\mathcal{D}\sim(1-\beta)^{\alpha-2}\to\infty$; and if $2<\alpha<4$, then $\mathcal{D}$ is finite, but $\mathcal{B}\sim(1-\beta)^{\alpha-4}\to\infty$.

IV. CONCLUDING REMARKS

Let us end with a comment on a recent study of the momentum diffusion in the Hamiltonian standard map [5]. These authors defined a function $\Psi(q)$ that is related to $P(q)$ simply as $P(q)=\langle v\rangle q-\Psi(q)$. It was found numerically that in the presence of an acceleration mode (corresponding to ballistic motion), the function $P(q)$ is the same as that of the acceleration-mode periodic orbit itself. On the other hand, in the presence of nonacceleration-mode islands (corresponding to stagnant motion) $P(q)$ is nontrivial and the diffusion coefficient $\mathcal{D}$ is finite. These results are in perfect agreement with our analytic predictions. On the basis of a similarity between the PM intermittency and two-dimensional Hamiltonian maps [13], we suggest that with nonaccelerator-mode islands, the function $P(q)$ of the standard map is not analytic; indeed, the Burnett coefficient $\mathcal{B}$ may already be divergent; and with accelerator modes the function $P(\beta,q)$ instead of $P(q)$ ought to be considered. Further studies are worthwhile from the present point of view, on the standard map as well as on more difficult problems like the stochastic web in the *ABC* flow [14].

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APPENDIX

In this Appendix we provide a proof of Eq. (13). If the unit interval is partitioned into three cells I_{-1}, I_0, and I_{+1}, each orbit of the map $f(x)$ [Fig. 1(c)] is uniquely coded by a string of spins: $x=(\sigma_0,\sigma_1,\ldots,\sigma_{n-1},\sigma_n,\ldots)$, if $f^i(x)\in I_{\sigma_i}$, $\sigma_i\equiv -1$, 0, or $+1$. The partition function $Z_n(\beta,q)$ [Eq. (4b)] is that of a spin lattice of size n; the interaction Hamiltonian of the n spins among themselves and the interaction of the spins with the external field, respectively, are represented by the first and second terms of the argument of the exponential function in Eq. (4b). The periodic boundary condition on the spin lattice is equivalent to the use of periodic orbits of the original dynamical system.

In order to evaluate

$$\ln\zeta(z,\beta,q)=\sum_{n=1}^{\infty}\frac{z^n}{n}Z_n(\beta,q)\ , \tag{A1}$$

we shall substitute the sum over spin states by a sum over states of *spin domains.* A k domain of type $i=-1,0,1$ consists of k spins of that type, necessarily followed by a spin of *different* type. Domains of type 0 are exclusively responsible for long-range interactions. We denote by u_n the interaction potential of an n domain of type 0,

$$u_n=\sum_{k=2}^{n+1}\lambda_k=\sum_{k=2}^{n+1}\ln\left[\frac{\Delta_{k-1}}{\Delta_k}\right]=\ln\left[\frac{\Delta_1}{\Delta_{n+1}}\right]. \tag{A2}$$

For the three types of spin domains, we define U_0, U_+, and U_- as follows:

$$U_0=\sum_{n=1}^{\infty}z^n\exp(-\beta u_n)=\sum_{n=1}^{\infty}z^n\left[\frac{\Delta_{n+1}}{\Delta_1}\right]^{\beta}, \tag{A3}$$

$$U_+=\sum_{n=1}^{\infty}z^n\exp(-\beta\lambda_1 n+qn)=\frac{1}{z^{-1}\exp(\beta\lambda_1-q)-1}, \tag{A4}$$

$$U_-=\sum_{n=1}^{\infty}z^n\exp(-\beta\lambda_1 n-qn)=\frac{1}{z^{-1}\exp(\beta\lambda_1+q)-1}, \tag{A5}$$

where $\lambda_1=\ln(1/\Delta_1)$.

Equation (A1) may now be expressed as a sum over all possible configurations of U_+, U_0, and U_- domains. In order to take into account the fact that one domain must be followed by another of different type, we define the following transfer matrix:

$$\mathbf{T}=\begin{bmatrix}0 & \sqrt{U_+U_0} & \sqrt{U_+U_-}\\ \sqrt{U_0U_+} & 0 & \sqrt{U_0U_-}\\ \sqrt{U_-U_+} & \sqrt{U_-U_0} & 0\end{bmatrix}. \tag{A6}$$

Then, it can be readily seen that $\ln[\zeta(z,\beta,q)]$ of Eq. (A1) with the periodic boundary condition may be written as

$$\ln[\zeta(z,\beta,q)]=\sum_{n=1}^{\infty}\left[\frac{z^n}{n}+\frac{z^n}{n}e^{-\beta\lambda_1 n+qn}+\frac{z^n}{n}e^{-\beta\lambda_1 n-qn}\right]+\sum_{d=1}^{\infty}\frac{1}{d}\mathrm{tr}(\mathbf{T}^d)\ , \tag{A7}$$

with $\mathrm{tr}(\mathbf{T})=0$.

The factor $1/d$ in Eq. (A7) is explained as follows. In the sum over spin states in $Z_n(\beta,q)$, there is a rotational symmetry that gives n times the value of a typical term. We may use this n to cancel the factor $1/n$ in Eq. (A1). The final result is the sum over the *spin states* that does not have the rotational symmetry. The corresponding sum over the *domain states* also breaks the rotational symmetry: a factor $1/d$ is included to compensate for the multiplicity d due to the rotational symmetry in $\mathrm{tr}(\mathbf{T}^d)$.

It follows from Eq. (A7) that

$$\zeta(z,\beta,q)=(1-z)^{-1}(1-ze^{-\beta\lambda_1+q})^{-1}\times(1-ze^{-\beta\lambda_1-q})^{-1}[\det(\mathbf{I}-\mathbf{T})]^{-1}\ , \tag{A8}$$

where $\mathbf{I}$ is a 3×3 identity matrix. An explicit calculation of $\det(\mathbf{I}-\mathbf{T})$, using Eqs. (A2)–(A6), yields Eq. (13).

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[10] Note that n in the sum of Eq. (15) is the length of a localized orbit. If we truncate it by the finite time t of observation, $n=t$, then a time-dependent diffusion coefficient $\mathcal{D}(t)$ is obtained. In this way, the following known results are recovered:

$$\langle[\mathbf{r}(t)-\mathbf{r}(0)]^2\rangle\sim\begin{cases}t & \text{if } \alpha>1\\ t/\ln t & \text{if } \alpha=1\\ t^{\alpha} & \text{if } \alpha<1\ .\end{cases}$$

[11] Again, if we truncate the sums in Eq. (20) at $n=t$, a time-dependent diffusion coefficient $\mathcal{D}(t)$ is obtained, and the following known results are recovered:

$$\langle [\mathbf{r}(t)-\mathbf{r}(0)]^2 \rangle \sim \begin{cases} t & \text{if } \alpha > 2 \\ t \ln t & \text{if } \alpha = 2 \\ t^{3-\alpha} & \text{if } 1 < \alpha < 2 \\ t^2/\ln t & \text{if } \alpha = 1 \\ t^2 & \text{if } \alpha < 1 \ . \end{cases}$$

[12] This point will be further discussed in X.-J. Wang (unpublished).

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